

Common zeros of inward vector fields on surfaces

Morris W. Hirsch*

Mathematics Department

University of Wisconsin at Madison

University of California at Berkeley

March 2, 2013

Abstract

A vector field X on a manifold M with possibly nonempty boundary is *inward* if it generates a unique local semiflow Φ^X . A compact relatively open set K in the zero set $Z(X)$ is a *block*. The Poincaré-Hopf index is extended to blocks $K \subset Z(X)$ where X is inward and K may meet ∂M . A block with nonzero index is *essential*.

Let X, Y be inward C^1 vector fields on surface M such that $[X, Y] \wedge X = 0$ and let K be an essential block of zeros for X . Among the main results are that Y has a zero in K if X and Y are analytic, or Y is C^2 and Φ^Y preserves area. Applications are made to actions of Lie algebras and groups.

Contents

1	Introduction	2
	Terminology	3
	Statement of results	4
	Background on group actions	7
2	Dynamics	7
3	Index functions	9
	The fixed point index for maps	9
	The equilibrium index for local semiflows	11
	The vector field index for inward vector fields	11
	Stability of essential blocks	13

*This research was supported in part by National Science Foundation grant DMS-9802182.

4 Proofs of the main theorems	14
Proof of Theorem 1.5	14
Proof of Theorem 1.7	17
Proof of Theorem 1.9	17
Proof of Theorem 1.10	18
Proof of Theorem 1.11	19

1 Introduction

Let M be an n -dimensional manifold with boundary ∂M and X a vector field on M , whose value at p is denoted by X_p . The *zero set* of X is

$$Z(X) := \{p \in M : X_p = 0\}.$$

The set of common zeros for a set \mathfrak{s} of vector fields is

$$Z(\mathfrak{s}) := \bigcap_{X \in \mathfrak{s}} Z(X).$$

A *block* for X is a compact, relatively open subset $K \subset Z(X)$. This means K lies in a precompact open set $U \subset M$ whose topological boundary $\text{bd}(U)$ contains no zeros of X . We say that U is *isolating* for (X, K) , and for X when $K := Z(X) \cap U$. When M is compact, $Z(X)$ is a block for X with M as an isolating neighborhood.

Definition 1.1 If $p \in M \setminus \partial M$ is an isolated zero of X , the *index of X at p* , denoted by $i_p X$, is the degree of the map of the unit $(n - 1)$ -sphere

$$\mathbf{S}^{n-1} \rightarrow \mathbf{S}^{n-1}, \quad z \mapsto \frac{\hat{X}(z)}{\|\hat{X}(z)\|},$$

where \hat{X} is the representative of X in an arbitrary coordinate system centered at p . When U is isolating for (X, K) and disjoint from $Z(X) \cap \partial M$, the *Poincaré-Hopf index* of X at K is

$$i_K^{\text{PH}}(X) := \sum_p i_p Y, \quad (p \in Z(Y) \cap U)$$

where Y is any vector field on M such that $Z(Y) \cap \overline{U}$ is finite, and there is a homotopy of vector fields $\{X^t\}_{0 \leq t \leq 1}$ from $X^0 = X$ to $X^1 = Y$ such that $\bigcup_t Z(X^t) \cap U$ is compact.¹

The block K is *essential* for X if $i_K^{\text{PH}}(X) \neq 0$.

Christian Bonatti [3] proved the following remarkable result:

¹The Poincaré-Hopf index goes back to Poincaré [28] and Hopf [15]. It is usually defined only when M is compact and $K = Z(X)$. The more general definition above is adapted from Bonatti [3].

Theorem 1.2 (BONATTI) *Assume $\dim M \leq 4$ and $\partial M = \emptyset$. If X, Y are analytic vector fields on M such that $[X, Y] = 0$, then $Z(Y)$ meets every essential block of zeros for X .²*

Our main results, Theorems 1.5 and 1.7, reach similar conclusions for surfaces M which may have nonsmooth boundaries, and certain pairs of vector fields that generate local semiflows on M , including cases where the fields are not analytic and do not commute. Applications are made to actions of Lie algebras and Lie groups.

Next we define terms (postponing some details), state the main theorems and apply them to Lie actions. After sections on dynamics and index functions, the main results are proved in Section 4.

Terminology

\mathbb{Z} denotes the integers, \mathbb{N}_+ the positive integers, \mathbb{R} the real numbers, and \mathbb{R}_+ the closed half line $[0, \infty)$. Maps are continuous, and manifolds are real, smooth and metrizable, unless otherwise noted. The set of fixed points of a map f is $\text{Fix}(f)$.

The following assumptions are always in force:

Hypothesis 1.3 *\tilde{M} is an analytic n -manifold with empty boundary. $M \subset \tilde{M}$ is a connected topological n -manifold.³*

We call M an *analytic manifold* when ∂M is an analytic submanifold of \tilde{M} . The tangent vector bundle of \tilde{M} is $T\tilde{M}$, whose fibre over p is the vector space $T_p\tilde{M}$. The restriction of $T\tilde{M}$ to a subset $S \subset \tilde{M}$ is the vector bundle $T_S\tilde{M}$. We set $TM := T_M\tilde{M}$.

A map f sending a set $S \subset M$ into a smooth manifold N is called C^r if it extends to a map \tilde{f} , defined on an open subset of \tilde{M} , that is C^r in the usual sense. Here $r \in \mathbb{N}_+ \cup \{\infty, \omega\}$, and C^ω means analytic. If f is C^1 and S has dense interior in M , the tangent map $T\tilde{f}: T\tilde{M} \rightarrow TN$ restricts to a bundle map $Tf: T_S M \rightarrow TN$ determined by f .

A *vector field* on S is a section $X: S \rightarrow T_S M$, whose value at p is denoted by X_p . The set of these vector fields is a linear space $\mathcal{V}(S)$. The linear subspaces

²Bonatti assumes $\dim M = 3$ or 4 , but the conclusion for $\dim M \leq 2$ follows easily: If $\dim M = 2$, identify M with $M \times \{0\} \subset M \times \mathbb{R}$ and apply Bonatti's theorem to the vector fields $(X, x \frac{\partial}{\partial x})$ and $(Y, x \frac{\partial}{\partial x})$ on $M \times \mathbb{R}$. For $\dim M = 1$ there is a simple direct proof.

³The only role of \tilde{M} is to permit a simple definition of smooth maps on M . Its global topology is never used, and in any discussion \tilde{M} can be replaced by any smaller open neighborhood of M . If $n > 2$ then M might not be smoothable, as shown by a construction due to Kirby [17]: Let P be a nonsmoothable closed 4-manifold (Freedman [7]). Let $D \subset P$ be a compact 4-disk. Then $M := P \setminus \text{Int } D$ is not smoothable, for otherwise ∂M would be diffeomorphic to \mathbf{S}^3 and P could be smoothed by gluing \mathbf{D}^4 to M . Define \tilde{M} as the connected sum of P with a nontrivial \mathbf{S}^2 -bundle over \mathbf{S}^2 . Then \tilde{M} is smoothable and contains M (compare Friedl *et al.* [8]).

$\mathcal{V}^r(S)$ and $\mathcal{V}^L(M)$, comprising C^r and locally Lipschitz fields respectively, are given the compact-open topology (uniform convergence on compact sets).

X and Y always denote vector fields on M . When X is C^r , \tilde{X} denotes an extension of X to a C^r vector field on an open set $W \subset \tilde{M}$.

The *Lie bracket* of $X, Y \in \mathcal{V}^1(M)$ is the restriction to M of $[\tilde{X}, \tilde{Y}]$. This operation makes $\mathcal{V}^\omega(M)$ and $\mathcal{V}^\infty(M)$ into Lie algebras.

$X \wedge Y$ denotes the tensor field of exterior 2-vectors $p \mapsto X_p \wedge Y_p \in \Lambda^2(T_p M)$. Evidently $X \wedge Y = 0$ iff X_p and Y_p are linearly dependent at all $p \in M$.

Inward vector fields A tangent vector to M at p is *inward* if it is the tangent at p to a smooth curve in M through p . The set of inward vectors at p is $T_p^{\text{in}} M$. A vector field X is *inward* if $X(M) \subset T^{\text{in}}(M)$, and there is a unique local semiflow $\Phi^X := \{\Phi_t^X\}_{t \in \mathbb{R}_+}$ on M whose trajectories are the maximally defined solutions to the initial value problems

$$\frac{dy}{dt} = X(y), \quad y(0) = p, \quad p \in M, \quad t \geq 0.$$

The set of inward vector fields is $\mathcal{V}_{\text{in}}(M)$. When ∂M is a C^1 submanifold of \tilde{M} , it can be shown that X is inward iff $X(M) \subset T^{\text{in}}(M)$.

Define

$$\mathcal{V}_{\text{in}}^r(M) := \mathcal{V}_{\text{in}}(M) \cap \mathcal{V}^r(M), \quad \mathcal{V}_{\text{in}}^L(M) := \mathcal{V}_{\text{in}}(M) \cap \mathcal{V}^L(M).$$

Proposition 2.3 shows these sets are convex cones in $\mathcal{V}(M)$.

The vector field index and essential blocks of zeros Let K be a block of zeros for $X \in \mathcal{V}_{\text{in}}(M)$, and $U \subset M$ an isolating neighborhood for (X, K) . The *vector field index*

$$i_K(X) := i(X, U) \in \mathbb{Z}$$

is defined in Section 3 as the fixed point index of the map $\Phi_t^X|_U: U \rightarrow M$, for any $t > 0$ so small that the compact set \bar{U} lies in the domain of Φ_t^X .

The block K is *essential* (for X) when $i_K(X) \neq 0$. A version of the Poincaré-Hopf theorem implies K is essential if it is an attractor for Φ^X and has nonzero Euler characteristic $\chi(K)$.

Statement of results

In the next two theorems, besides Hypothesis 1.3 we assume:

Hypothesis 1.4

- M and \tilde{M} are surfaces,

- X and Y are C^1 inward vector fields on M ,
- $K \subset M$ is an essential block of zeros for X ,
- $[X, Y] \wedge X = 0$.

The last condition has the following dynamical significance (Proposition 2.2):

- Φ^Y permutes integral curves of X .

This implies:

- if $q = \Phi_t^Y(p)$ then $X_q = \lambda \cdot T\Phi_t^Y(X_p)$ for some $\lambda > 0$,
- $Z(X)$ is positively invariant under Φ^Y . (See Definition 2.1.)

A cycle for Y , or a Y -cycle, is a periodic orbit of Φ^Y that is not a fixed point.

Theorem 1.5 Assume Hypothesis 1.4. Each of the following conditions implies $Z(Y) \cap K \neq \emptyset$:

- (a) X and Y are analytic.
- (b) Every neighborhood of K contains an open neighborhood whose boundary is a nonempty union of finitely many Y -cycles.

When $[X, Y] = 0$ this extends Bonatti's Theorem to surfaces with nonempty boundaries. The case $[X, Y] = cX$, $c \in \mathbb{R}$ yields applications to actions of Lie algebras and Lie groups.

Example 1.6

In his pioneering paper [18], E. Lima constructs vector fields X, Y on the closed disk \mathbf{D}^2 , tangent to $\partial\mathbf{D}^2$ and generating unique flows, such that $[X, Y] = X$ and $Z(X) \cap Z(Y) = \emptyset$ (see Remark 4.2). Such fields can be C^∞ (M. Belliard & I. Liousse [2], F.-J. Turiel [33]). The unique block of zeros for X is $Z(X) = \partial\mathbf{D}^2$, which is essential because $\chi(\mathbf{D}^2) \neq \emptyset$, but $Z(Y)$ is a point in the interior of \mathbf{D}^2 . This shows that the conclusion of Theorem 1.5(a) can fail when X and Y are not analytic. The flows of X and Y generate an effective, fixed-point free action by a connected, solvable nonabelian Lie group.

A local semiflow on a surface M preserves area if it preserves a Borel measure on M that is positive and finite on nonempty precompact sets.

Theorem 1.7 Assume Hypothesis 1.4. If Φ^Y preserves area, each of the following conditions implies $Z(Y) \cap K \neq \emptyset$:

- (i) K contains a Y -cycle,

- (ii) Y is C^2 ,
- (iii) K has a planar neighborhood in M .

Definition 1.8 For $X \in \mathcal{V}_{\text{in}}^\omega(M)$ define

$$W(X) := \{Y \in \mathcal{V}_{\text{in}}^\omega(M) : [X, Y] \wedge X = 0\},$$

which is the set of inward analytic vector fields on M whose local semiflows permute integral curves of X . Propositions 2.2 and 2.3 imply $W(X)$ is a convex cone that is closed under Lie brackets, and a subalgebra of $\mathcal{V}^\omega(M)$ if M is an analytic manifold without boundary.

Theorem 1.9 *Assume Hypothesis 1.4 holds. If X is analytic and ∂M is an analytic subset of \tilde{M} , then $Z(W(X)) \cap K \neq \emptyset$.*

Actions of Lie algebras and Lie groups

Theorem 1.10 *Let M be an analytic surface and \mathfrak{g} a Lie algebra (perhaps infinite dimensional) of analytic vector fields on M that are tangent to ∂M . Assume $X \in \mathfrak{g}$ spans a nontrivial ideal. Then:*

- (a) $Z(\mathfrak{g})$ meets every essential block of zeros for X .
- (b) If M is compact and $\chi(M) \neq 0$, then $Z(\mathfrak{g}) \neq \emptyset$.

Note that \mathfrak{g} has 1-dimensional ideal if its center is nontrivial, or \mathfrak{g} is finite dimensional and supersoluble (Jacobson [16, Ch. 2, Th. 14]). A finite dimensional solvable Lie algebra of vector fields on a surface has derived length ≤ 3 (Epstein & Thurston [6]). Plante [26] points out that $\mathcal{V}^\omega(\mathbb{R}^2)$ contains infinite-dimensional subalgebras, such as the Lie algebra of quadratic vector fields.

An *action* of a group G on a manifold M is a homomorphism $\alpha: g \rightarrow g^\alpha$ from G to the homeomorphism group of P , such that the corresponding *evaluation map*

$$\text{ev}_\alpha: G \times P \rightarrow P, \quad (g, p) \mapsto g^\alpha(p)$$

is continuous. When ev_α is analytic, α is an *analytic action*.

Theorem 1.11 *Assume M is a compact analytic surface and G is a connected Lie group having a one-dimensional normal subgroup. If $\chi(M) \neq 0$, every effective analytic action of G on M has a fixed point.*

For supersoluble G this is due to Hirsch & Weinstein [14].

Background on group actions

The literature on actions of connected Lie groups G include the following notable results:

Proposition 1.12 *If G is solvable (respectively, nilpotent) and acts effectively on an n -dimensional manifold, the derived length of G is $\leq n + 1$ (respectively, $\leq n$) (Epstein & Thurston [6]).*

In the next two propositions M denotes a compact connected surface.

Proposition 1.13 *Assume G acts on M without fixed points.*

- (i) *If G is nilpotent, $\chi(M) \neq 0$ (Plante [25]).*
- (ii) *If the action is analytic, $\chi(M) \geq 0$ (Turiel [33], Hirsch [11]).*

Proposition 1.14 *Let $\text{Aff}_+(\mathbb{R}^m)$ denote the group of orientation-preserving affine homeomorphisms of \mathbb{R}^m .*

- (a) *If $\chi(M) < 0$ and G acts effectively on M without fixed points, then G has a quotient isomorphic to $\text{Aff}_+(\mathbb{R}^1)$ (Belliat [1]).*
- (b) *$\text{Aff}_+(\mathbb{R}^1)$ has effective fixed-point free actions on M (Plante [25]).*
- (c) *$\text{Aff}_+(\mathbb{R}^2)$ has effective analytic actions on M (Turiel [33]).*

For related results see the references above, also Belliat [1], Hirsch [12, 13], Molino & Turiel [21, 22], Plante [27], Thurston [31], Turiel [32]. Transitive effective surface actions are classified in Mostow's thesis [23], with a useful summary in Belliat [1].

2 Dynamics

Let $\Psi := \{\Psi_t\}_{t \in \mathbb{T}}$ denote a local flow ($\mathbb{T} = \mathbb{R}$) or a local semiflow ($\mathbb{T} = \mathbb{R}_+$) on a topological space S . Each Ψ_t is a homeomorphism from an open set $\mathcal{D}(\Psi_t) \subset S$ onto a set $\mathcal{R}(\Psi_t) \subset S$, such that:

- $\Psi_t(p)$ is continuous in (t, p) ,
- Ψ_0 is the identity map of S ,
- if $0 \leq |s| \leq |t|$ and $|st| \geq 0$ then $\mathcal{D}(\Psi^s) \supset \mathcal{D}(\Psi^t)$,
- $\Psi_t(\Psi_s(p)) = \Psi_{t+s}(p)$.

We adopt the convention that notation of the form “ $\Psi_t(x)$ ” presumes $x \in \mathcal{D}(\Psi_t)$.

Definition 2.1 A set $L \subset S$ is *positively invariant* under Ψ provided Ψ_t maps $L \cap \mathcal{D}(\Psi_t)$ into L for all $t \geq 0$, and *invariant* when $L \subset \mathcal{D}(\Psi_t) \cap \mathcal{R}(\Psi_t)$ for all $t \in \mathbb{T}$. When Ψ is generated by a vector field Y we use the analogous terms “positively Y -invariant” and “ Y -invariant.”

Let M, \tilde{M} be as in Hypothesis 1.3. When Ψ is a local semiflow on M , the theorem on invariance of domain shows that $\mathcal{R}(\Psi_t)$ is open in M when Φ is a local flow, and also when $\mathcal{D}(\Psi_t) \cap \partial M = \emptyset$. This implies $M \setminus \partial M$ is positively invariant under every local semiflow on M .

Proposition 2.2 Assume $X, Y \in \mathcal{V}_{\text{in}}(M)$ and $[X, Y] \wedge X = 0$.

- (a) If $\Phi_t^Y(p) = q$ then $T_p \Phi_t^Y : X_p \mapsto cX_q$, $c > 0$.
- (b) Φ_t^Y sends integral curves of X to integral curves of X .
- (c) $Z(X)$ is Y -invariant.

Proof Let \tilde{X} and $\tilde{Y} \in \mathcal{V}^1(\tilde{M})$ be extensions of X and Y , respectively. It suffices to prove:

(*) If $p \in M$, $t \geq 0$ and $p(t) := \Phi_t^{\tilde{X}}(p)$, then the linear map

$$T_p \Phi_t^{\tilde{Y}} : T_p \tilde{M} \rightarrow T_{p(t)} \tilde{M}$$

sends X_p to a positive scalar multiple of $X_{p(t)}$.

By continuity it suffices to prove this when $Y_p \neq 0$, $X_p \neq 0$ and $|t|$ sufficiently small. Working in flowbox coordinates (u_j) for \tilde{Y} in a neighborhood of p , we assume \tilde{M} is an open set in \mathbb{R}^n , $\tilde{Y} = \frac{\partial}{\partial u_1}$, and \tilde{X} has no zeros. Because $[X, Y] \wedge X = 0$, there is a unique continuous map $f : M \rightarrow \mathbb{R}$ such that $[X, Y] = fX$. Since \tilde{Y} is a constant vector field, the vector-valued function $t \mapsto X_{p(t)}$ satisfies

$$\frac{dX_{p(t)}}{dt} = -f(p(t)) \cdot X_{p(t)},$$

whose solution is

$$X_{p(t)} = e^{-\int_0^t f(s)ds} \cdot X_{p(0)}.$$

This implies (*). ■

The following fact is somewhat surprising because $T_p^{\text{in}} M$ need not be convex in $T_p M$:

Proposition 2.3 $\mathcal{V}_{\text{in}}^{\text{L}}(M)$ is a convex cone in $\mathcal{V}(M)$.

Proof As $\mathcal{V}^L(M)$ is a convex cone in $\mathcal{V}(M)$, it suffices to show that $\mathcal{V}_{\text{in}}^L(M)$ is closed under addition. Let $X, Y \in \mathcal{V}^L(M)$. We need to prove:

$$\text{If } p \in \partial M \text{ there exists } \epsilon > 0 \text{ such that } 0 \leq t \leq \epsilon \implies \Phi_t^{X+Y}(p) \in M. \quad (1)$$

This is easily reduced to a local result, hence we assume M is relatively open in the closed halfplane $\mathbb{R} \times [0, \infty)$ and X, Y are Lipschitz vector fields on M . Let \tilde{X}, \tilde{Y} be extensions of X, Y to Lipschitz vector fields on an open neighborhood $\tilde{M} \subset \mathbb{R}^2$ of M (Johnson *et al.* [19]). Denote the local flows of $\tilde{X}, \tilde{Y}, \tilde{X} + \tilde{Y}$ respectively by $\{f_t\}, \{g_t\}, \{h_t\}$, ($t \in \mathbb{R}$). We use a special case of Nelson [24, Th. 1, Sec. 4]:

Proposition 2.4 *For every $p \in \tilde{M}$ there exists $\epsilon > 0$ and a neighborhood $W \subset \tilde{M}$ of p such that*

$$h_t(x) = \lim_{k \rightarrow \infty} (f_{t/k} \circ g_{t/k})^k(x)$$

uniformly for $x \in W$ and $|t| < \epsilon$.

Because X and Y are inward, M is positively invariant under the local semiflows $\{f_t\}_{t \geq 0}$ and $\{g_t\}_{t \geq 0}$. Therefore

$$0 \leq t \leq \epsilon \implies (f_{t/k} \circ g_{t/k})^k \in M, \quad (k \in \mathbb{N}_+).$$

As M is relatively closed in \tilde{M} , Proposition 2.4 implies $h_t(x) \in M$ for $0 \leq t \leq \epsilon$, which yields (1). ■

Examination of the proof yields:

Corollary 2.5 *If L is a closed subset of a smooth manifold N , the set of locally Lipschitz vector fields on N for which L is positively invariant is a convex cone.* ■

Question 2.6 Is $\mathcal{V}_{\text{in}}(M)$ is a convex cone in $\mathcal{V}(M)$?

3 Index functions

We review properties of the fixed point index $I(f)$ defined by the late Professor Albrecht Dold ([4, 5]). Using it we define an *equilibrium index* $I_K(\Phi)$ for local semiflows, and a *vector field index* $i_K(X)$ for inward vector fields.

The fixed point index for maps

Dold's Hypothesis:

- V is an open set in a topological space S .
- $f: V \rightarrow S$ is a continuous map with compact fixed point set $\text{Fix}(f) \subset V$.

- V is a Euclidean neighborhood retract (ENR).⁴

On the class of maps satisfying these conditions, Dold constructs an integer-valued *fixed point index* denoted here by $I(f)$, uniquely characterized by the following five properties (see [5, VII.5.17, Ex. 5*]):

(FP1) $I(f) = I(f|V_0)$ if $V_0 \subset V$ is an open neighborhood of $\text{Fix}(f)$.

(FP2) $I(f) = \begin{cases} 0 & \text{if } \text{Fix}(f) = \emptyset, \\ 1 & \text{if } f \text{ is constant.} \end{cases}$

(FP3) $I(f) = \sum_{i=1}^m I(f|V_i)$ if V is the disjoint union of m open sets V_i .

(FP4) $I(f \times g) = I(f) \cdot I(g)$.

(FP5) $I(f_0) = I(f_1)$ if there is a homotopy $f_t: V \rightarrow S$, $(0 \leq t \leq 1)$ such that $\bigcup_t \text{Fix}(f_t)$ is compact.

These correspond to (5.5.11) — (5.5.15) in [5, Chap. VII].

In addition:

(FP6) If f is C^1 and $\text{Fix}(f)$ is an isolated fixed point p , then

$$I(f) = (-1)^\nu$$

where ν is the number of eigenvalues λ of f such that $\lambda > 1$, ignoring multiplicities ([5, VII.5.17, Ex. 3]).

(FP7) If S is an ENR and $f: S \rightarrow S$ is homotopic to the identity map, then

$$I(f) = \chi(S).$$

See Dold [5, VII.6.22].

Lemma 3.1 *If g is sufficiently close to f in the compact open topology, then $\text{Fix}(g)$ is compact and $I(g) = I(f)$.*

Proof We can assume $\rho: W \rightarrow V$ is a retraction, where $W \subset \mathbb{R}^m$ is an open set containing V . For g sufficiently close to f the following hold: W contains the line segment (or point) spanned by $\{f(p), g(p)\}$ for every $p \in V$, and the maps

$$f_t: (t, p) \mapsto \rho((1-t)f(p) + tg(p)), \quad (0 \leq t \leq 1, \quad p \in V)$$

constitute a homotopy in V from $f_0 = f$ to $f_1 = g$ with $\bigcup_t \text{Fix}(f_t)$ is compact. Therefore the conclusion follows from (FP5). ■

⁴This means V is homeomorphic to a retract of an open subset of some Euclidean space. Polyhedra and connected metrizable manifolds are ENRs.

The equilibrium index for local semiflows

Let $\Phi := \{\Phi_t\}_{t \geq 0}$ be a local semiflow in a topological space C , with *equilibrium set*

$$\mathcal{E}(\Phi) := \bigcap_{t \geq 0} \text{Fix}(\Phi_t).$$

$K \subset \mathcal{E}(\Phi)$ is a *block* if K is compact and has an open, precompact ENR neighborhood $V \subset C$ such that $\overline{V} \cap \mathcal{E}(\Phi) \subset V$. Such a V is an *isolating neighborhood* for K . With these assumptions on Φ and V we have:

Lemma 3.2 *There exists $\tau > 0$ such that the following hold when $0 < t \leq \tau$:*

- (a) $\text{Fix}(\Phi_t) \cap V$ is compact,
- (b) $I(\Phi_t|V) = I(\Phi_\tau|V)$.

Proof If (a) fails, there are convergent sequences $\{t_k\}$ in $[0, \infty)$ and $\{p_k\}$ in V such that

$$t_k \searrow 0, \quad p_k \in \text{Fix}(\Phi_{t_k}) \cap V, \quad p_k \rightarrow q \in \text{bd}(V).$$

Joint continuity of $(t, x) \mapsto \Phi_t(x)$ yields the contradiction $q \in \mathcal{E}(\Phi) \cap \text{bd}(V)$. Assertion (b) is a consequence of (a) and (FP5). ■

It follows that the fixed point index $I(\Phi_\tau|V)$ depends only on Φ and K , and is the same for all isolating neighborhoods V of K .

Definition 3.3 Let $\tau > 0$ be as in Lemma 3.2(b). We call $I(\Phi_\tau|V)$ the *equilibrium index* of Φ in V , and at K , denoted by $i(\Phi, V)$ and $i_K(\Phi)$.

The vector field index for inward vector fields

In the rest of this section the manifolds \tilde{M} and $M \subset \tilde{M}$ are as in Hypothesis 1.3, K is a block of zeros for X for $X \in \mathcal{V}_{\text{in}}(M)$, and U an isolating neighborhood for (X, K) . Then K is also a block of equilibria for the local semiflow Φ^X , and the equilibrium index $i(\Phi^X, U)$ is defined in Definition 3.3.

Definition 3.4 The *vector field index* of X in U (and at K) is

$$i(X, U) = i_K(X) := i(\Phi^X, U).$$

K is *essential* (for X) if $i_K(X) \neq 0$, and *inessential* otherwise.

Two vector fields $X_j \in \mathcal{V}_{\text{in}}(M_j)$, $j = 1, 2$ have *isomorphic germs* at $K_j \subset M_j$ provided there are open neighborhoods $U_j \subset M_j$ of K_j and a homeomorphism $U_1 \approx U_2$ conjugating $\Phi^{X_1}|_{U_1}$ to $\Phi^{X_2}|_{U_2}$.

Proposition 3.5 *The vector field index has the following properties:*

(VF1) $i_K(X) = i_{K'}(X')$ if K' is a block of zeros for $X' \in \mathcal{V}_{\text{in}}(M')$ and the germs of X at K and X' at K' are isomorphic.

(VF2) If $i(X, U) \neq 0$ then $Z(X) \cap U \neq \emptyset$.

(VF3) $i(X, U) = \sum_{j=1}^m i(X, U_j)$ provided U is the union of disjoint open sets U_1, \dots, U_m .

(VF4) If Y is sufficiently close to X in $\mathcal{V}_{\text{in}}(M)$, then U is isolating for Y and $i(X, U) = i(Y, U)$.

(VF5) $i_K(X)$ equals the Poincaré-Hopf index $i_K^{\text{PH}}(X)$ provided $K \cap \partial M = \emptyset$.

Proof

(VF1): A consequence of (FP1).

(VF2): Follows from (FP2).

(VF3): Follows from (FP3).

(VF4): Use Lemma 3.1.

(VF5): Since X can be approximated by locally C^∞ vector fields transverse to the zero section, using compactness of \overline{U} and (VF4) we assume $Z(X) \cap U$ is finite set of hyperbolic equilibria. By (FP3) we assume $Z(X) \cap U$ is a hyperbolic equilibrium p . In this case the index of X at p is $(-1)^\nu$ where ν is the number of positive eigenvalues of dX_p (ignoring multiplicity). The conclusion follows from (FP6) and Definitions 3.3, 3.4. ■

Proposition 3.6 *If M is compact, $i(X, M) = \chi(M)$.*

Proof Follows from (FP7). ■

Proposition 3.7 *Assume $X, Y \in \mathcal{V}_{\text{in}}(M)$ and $U \subset M$ is isolating for X . Then U is isolating for Y , and*

$$i(X, U) = i(Y, U),$$

provided one of the following holds:

(i) $Y|_{\text{bd}(U)}$ is sufficiently close to $X|_{\text{bd}(U)}$,

(ii) $Y|_{\text{bd}(U)}$ is nonsingularly homotopic to $X|_{\text{bd}(U)}$.

Proof Both (i) and (ii) imply U is isolating for Y . Consider the homotopy

$$Z^t := (1 - t)X + tY, \quad (0 \leq t \leq 1).$$

When (i) holds each vector field $Z^t|_{\text{bd}(U)}$ is nonsingular, implying (ii). In addition, Z^t is inward by Proposition 2.3, and Lemma 3.2 yields $\tau > 0$ such that

$$0 < t \leq \tau \implies i(X, U) = I(\Phi_t^X|U), \quad i(Y, U) = I(\Phi_t^Y|U).$$

By Lemma 3.1, each $t \in [0, 1]$ has an open neighborhood $J_t \subset [0, 1]$ such that

$$s \in J_t \implies I(\Phi_\tau^{X^s}|U) = I(\Phi_\tau^{X^t}|U).$$

Covering $[0, 1]$ with sets J_{t_1}, \dots, J_{t_v} and inducting on $v \in \mathbb{N}_+$ shows that

$$I(\Phi_\tau^X|U) = I(\Phi_\tau^Y|U),$$

which by Definition 3.4 implies the conclusion. ■

The index as an obstruction The following algebraic calculation of the index is included for completeness, but not used. Assume M is oriented and U is an isolating neighborhood for a block $K \subset Z(X)$. Let $V \subset U$ be a compact smooth n -manifold with the orientation induced from M , such that $K \subset V \setminus \partial V$. The primary obstruction to extending $X|_{\partial V}$ to a nonsingular section of TV is the *relative Euler class*

$$\mathbf{e}_{(X,V)} \in H^n(V, \partial V)$$

Let

$$\mathbf{v} \in H_n(V, \partial V)$$

be the homology class corresponding to the induced orientation of V . Denote by

$$H^n(V, \partial V) \times H_n(V, \partial V) \rightarrow \mathbb{Z}, \quad (\mathbf{c}, \mathbf{u}) \rightarrow \langle \mathbf{c}, \mathbf{u} \rangle,$$

the Kronecker Index pairing, induced by evaluating cocycles on cycles. Unwinding definitions leads to:

Proposition 3.8 *With M, X, K, V as above,*

$$i_K(X) = \langle \mathbf{v}, \mathbf{e}_{(X,V)} \rangle.$$

When M is nonorientable the same formula holds provided the coefficients for $H^n(V, \partial V)$ and $H_n(V, \partial V)$ are twisted by the orientation sheaf of V .

Stability of essential blocks

An immediate consequence of Propositions 3.7(i) and property (VF2) of 3.5 is:

Corollary 3.9 *If a block K is essential for X , and $Y \in \mathcal{V}_{\text{in}}(M)$ is sufficiently close to X , then every neighborhood of K contains an essential block for Y .* ■

Thus essential blocks are stable under perturbations of the vector field. It is easy to see that a block is stable if it contains a stable block. For example, the block $\{-1, 1\}$ for $X = (x^2 - 1)\frac{\partial}{\partial x}$ on \mathbb{R} is stable, but inessential. But the following result (not used) means that a block can be perturbed away iff every subblock is inessential:

Proposition 3.10 *Assume ∂M is a smooth submanifold of \tilde{M} and every block for X in U is inessential. Then $X = \lim_{k \rightarrow \infty} X^k$ where*

$$X = \lim_{k \rightarrow \infty} X^k, \quad X^k \in \mathcal{V}_{\text{in}}^{\text{L}}(M), \quad Z(X^k) \cap U = \emptyset, \quad (k \in \mathbb{N}_+)$$

and X^k coincides with X outside U .

Proof Fix a Riemannian metric on M . For every $\epsilon > 0$ choose an isolating neighborhood $W := W(\epsilon) \subset U$ of K having only finitely many components, and such that

$$\|X_p\| < \epsilon, \quad (p \in W).$$

Thus $X(W)$ lies in the bundle $T^\epsilon W$ whose fibre over p is the open disk of radius ϵ in $T_p W$. Smoothness of ∂M enables an approximation $Y^\epsilon \in \mathcal{V}_{\text{in}}^{\text{L}}$ to X such that $Y^\epsilon(W) \subset T^\epsilon W$ and $Z(Y^\epsilon) \cap \overline{U}$ is finite. By Proposition 3.7(ii) and the hypothesis we choose the approximation close enough so that for each component W_j of W :

$$i(Y^\epsilon, W_j) = 0.$$

Standard deformation techniques (compare Hirsch [10, Th. 5.2.10]) permit pairwise cancellation in each W_j of the zeros of Y^ϵ , without changing Y^ϵ near $\text{bd}(W_j)$. This yields a vector field $X^\epsilon \in \mathcal{V}_{\text{in}}^{\text{L}}(M)$ coinciding with X in a neighborhood of $M \setminus W$ and nonsingular in W , and such that

$$\|X_p^\epsilon - X_p\| < 2\epsilon, \quad (p \in M)$$

The sequence $\{X^{1/k}\}_{k \in \mathbb{N}_+}$ has the required properties. ■

Plante [27] discusses index functions for abelian Lie algebras of vector fields on closed surfaces.

4 Proofs of the main theorems

Proof of Theorem 1.5

We recall the hypothesis:

- \tilde{M} is an analytic surface with empty boundary, $M \subset \tilde{M}$ is a connected topological surface embedded in \tilde{M} .
- X and Y are inward C^1 vector fields on M .
- $[X, Y] \wedge X = 0$.
- K is an essential block of zeros for X .

Definition 4.1 Let A, B be vector fields on a set $S \subset \tilde{M}$. The *dependency set* of A and B is

$$D(A, B) := \{p \in S : A_p \wedge B_p = 0\}$$

Evidently

$$D(X, Y) = D(\tilde{X}, \tilde{Y}) \cap M.$$

Proposition 2.2 implies $D(\tilde{X}, \tilde{Y})$ is \tilde{Y} -invariant and $D(X, Y)$ is positively Y -invariant.

Case (a): X and Y are analytic. Then $D(\tilde{X}, \tilde{Y})$ and its subset $Z(\tilde{X}, \tilde{Y})$ are analytic sets in \tilde{M} , hence \tilde{M} is a simplicial complex with subcomplexes $D(\tilde{X}, \tilde{Y})$ and $Z(\tilde{X}, \tilde{Y})$ by S. Łojasiewicz's triangulation theorem [20].

We assume

- $\dim Z(\tilde{Y}) < 2$,

as otherwise $Y = 0$ and the conclusion is trivial. We also assume

- *every component of K has dimension 1*

because isolated points of K lie in $Z(Y)$ and $K = M$ by analyticity if some component of K is 2-dimensional, and either of these conditions imply the conclusion. Thus $\Psi^{\tilde{Y}}$ restricts to a semiflow on the 1-dimensional complex $D(\tilde{X}, \tilde{Y})$ having $Z(X)$ and $D(X, Y)$ as positively invariant subcomplexes.

Let $J \subset K$ be any component. J is a compact, connected, triangulable space of dimension ≤ 1 which is positively Y -invariant. From the topology of J we see that $Z(Y)$ meets J and therefore K , unless

$$J \text{ is a Jordan curve on which } \Phi^Y \text{ acts transitively.} \quad (2)$$

Henceforth (2) is assumed.

Let $L \subset D(X, Y)$ be the component containing J . The set $Q := J \cap \overline{L \setminus J}$ is positively Y -invariant, whence (2) implies $Q = J$ or $Q = \emptyset$. Therefore one of the following holds:

(D1) $J \subset \text{Int}_M D(X, Y)$,

(D2) J is a component of $D(X, Y)$.

Assume (D1) and suppose *per contra* that $Z(Y) \cap K = \emptyset$. Then $D(X, Y)$ contains the compact closure of an open set U that is isolating for (X, K) . We choose U so that each component C of the topological boundary $\text{bd}(U)$ is a Jordan curve or a compact arc. It suffices by Proposition 3.7(ii) to prove for each C :

$$\text{the vector fields } X|_C \text{ and } Y|_C \text{ are nonsingularly homotopic.} \quad (3)$$

Since this holds when C is an arc, we assume C is a Jordan curve. Fix a Riemannian metric on M and define

$$\hat{X}_p = \frac{1}{\|X_p\|} X_p, \quad \hat{Y}_p = \frac{1}{\|Y_p\|} Y_p, \quad (p \in C).$$

These unit vector fields are nonsingularly homotopic to $X|_C$ and $Y|_C$ respectively, and the assumption $C \subset D(X, Y)$ implies $\hat{X} = \hat{Y}$ or $\hat{X} = -\hat{Y}$. In the first case there is nothing more to prove. In the second case \hat{X} and \hat{Y} are antipodal sections of the unit circle bundle η associated to $T_C M$. As the identity and antipodal maps of the circle are homotopic through rotations, (3) is proved.

Now assume (D2). There is an isolating neighborhood U for X such that

$$U \cap D(X, Y) = K. \quad (4)$$

If $0 < \epsilon < 1$ the field $X^\epsilon := (1 - \epsilon)X + \epsilon Y$ belongs to $\mathcal{V}_{\text{in}}^L(M)$ (Proposition 2.3), and has a zero $p \in U$ provided ϵ is sufficiently small (Proposition 3.9). In that case X_p and Y_p are linearly dependent, therefore $p \in K$ by (4), whence $Y_p = 0$.

Case (b): Every neighborhood of K contains an open neighborhood W whose boundary consist of finitely many Y -cycles. It suffices to prove that $Z(Y) \cap \overline{W} \neq \emptyset$ if W is isolating for (X, K) . Given such a W , let C be a component of $\text{bd}(W)$. By Proposition 2.2(a), X_p and Y_p are linearly dependent at all points of C , or at no point of C . In the first case $X|_C$ and $Y|_C$ are nonsingularly homotopic, as in the proof of (3). In the second case they are nonsingularly homotopic by the restriction to C of the path of vector fields $(1 - t)X + tY$, $0 \leq t \leq 1$. It follows that $X|_{\text{bd}(W)}$ and $Y|_{\text{bd}(W)}$ are nonsingularly homotopic. Now Proposition 3.7(ii) implies

$$i(Y, W) = i(X, W),$$

which is nonzero because K is essential for X . Hence either $Z(Y)$ meets $\text{bd}(W)$, or W is isolating for (Y, K) and $Z(Y) \cap W \neq \emptyset$. This shows that $Z(Y)$ meets \overline{W} . ■

Remark 4.2 It is interesting to see where the proof Theorem 1.5 breaks down in Lima's counterexample to a nonanalytic version (see Example 1.6). Lima starts from the planar vector fields

$$X^1 := \frac{\partial}{\partial x}, \quad Y^1 := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

satisfying $[X^1, Y^1] = X^1$ and transfers them to the open disk by an analytic diffeomorphism $f: \mathbb{R}^2 \approx \text{Int } \mathbf{D}^2$. This is done in such a way that the push-forwards of X^1 and Y^1 extend to continuous vector fields X, Y on $M := \mathbf{D}^2$ satisfying $[X, Y] = Y$, with $Z(X) = K = \partial \mathbf{D}^2$ and $i_K(X) = 1$, while $Z(Y)$ is a singleton in the interior of \mathbf{D}^2 . This can be done so that X and Y are C^∞ (see [2]) and therefore

generate unique local semiflows. The dependency set $D(X, Y)$ is $R \cup \partial D^2$, where R is the Φ^X -orbit of z , a topological line in $\text{Int } D^2$ that spirals toward the boundary in both direction. $D(X, Y)$ is not triangulable because it is connected but not path connected. It is easily seen that neither (D1) nor (D2) holds.

Proof of Theorem 1.7

Here K is essential for X and Φ^Y preserves area. Suppose *per contra*

$$Z(Y) \cap K = \emptyset. \quad (5)$$

We can assume K contains a Y -cycle γ , for (5) implies every minimal set for Φ^Y in K is a cycle: This follows from the Poincaré-Bendixson theorem (Hartman [9]) when K has a planar neighborhood, and from the Schwartz-Sacksteder Theorem [30, 29] when Y is C^2 .

Let $J \subset M$ be a half-open arc with endpoint $p \in \gamma$ and otherwise topologically transverse to Y orbits (Whitney [35]). For any sufficiently small half-open subarc $J_0 \subset J$ with endpoint p , there is a first-return Poincaré map $f: J_0 \hookrightarrow J$ obtained by following trajectories. By the area-preserving hypothesis and Fubini's Theorem, f is the identity map of J_0 . Therefore γ has a neighborhood $U \subset M$, homeomorphic to a cylinder or a Möbius band, filled with Y -cycles. Theorem 1.5(b) implies K is inessential for X , contradicting to the hypothesis. ■

Proof of Theorem 1.9

K is an essential block for $X \in \mathcal{V}_{\text{in}}^\omega(M)$ and ∂M is an analytic set in \tilde{M} . We can assume $W(X) \neq \emptyset$ (see Definition 1.8). Our goal is to prove

$$Z(W(X)) \cap K \neq \emptyset.$$

The main step is to show that the set

$$\mathcal{P}(K) := \{t \subset W(X) : Z(t) \cap K \neq \emptyset\}$$

is inductively ordered by inclusion. Note that $\mathcal{P}(K)$ is nonempty because it contains the singleton $\{Y\}$. In fact Theorem 1.5 states:

$$Y \in W(X) \implies \{Y\} \in \mathcal{P}(K). \quad (6)$$

We rely on a consequence of Proposition 2.2:

$$K \text{ is positively invariant under } W(X) \quad (7)$$

The assumption on ∂M implies M is semianalytic as a subset of \tilde{M} , and this implies K is also semianalytic. Therefore K , being compact, has only finitely

many components and one of them is essential for X by Proposition 3.5, (VF3). Therefore we can assume K is a connected semianalytic set of \tilde{M} . Now K is the intersection of M with the component of $Z(\tilde{X})$ that contains K , which is an analytic set. This implies $\dim K \leq 1$, and we assume $\dim K = 1$, as otherwise K is finite and contained in $Z(W(X))$ by (7).

The set $K_{sing} \subset K$ where K is not locally an analytic 1-manifold is finite and positively invariant under $W(X)$. As this implies $K_{sing} \subset Z(W(X)) \cap K$, we can assume $K_{sing} = \emptyset$, which under current assumptions means:

K is an analytic submanifold of \tilde{M} diffeomorphic to a circle.

We can also assume:

(K) $K \subset L$ if $K \cap L \neq \emptyset$ and L is positively $W(X)$ -invariant semianalytic set in \tilde{M} .

For if $Y \in W(S)$ then $K \cap L$, being nonempty, finite and positively invariant under every Y , is necessarily contained in $Z(Y)$.

From (K) we infer

$$t \in \mathcal{P}(K) \implies K \subset Z(t). \quad (8)$$

Consequently $\mathcal{P}(K)$ is inductively ordered by inclusion. By Zorn's lemma there is a maximal element $\mathfrak{m} \in \mathcal{P}(K)$, and (8) implies

$$K \subset Z(\mathfrak{m}). \quad (9)$$

To prove every $Y \in \mathfrak{g}$ lies in \mathfrak{m} , let $\mathfrak{n}_Y \subset \mathfrak{g}$ be the smallest ideal containing Y and \mathfrak{m} . Theorem 1.5 implies $Z(Y) \cap K \neq \emptyset$, whence

$$Z(\mathfrak{n}_Y) \cap K = Z(Y) \cap Z(\mathfrak{m}) \cap K \neq \emptyset$$

by (9). Property (K) shows that $\mathfrak{n}_Y \in \mathcal{H}$, so $\mathfrak{n}_Y = \mathfrak{m}$ by maximality of \mathfrak{m} . ■

Proof of Theorem 1.10

The theorem states:

Let M be an analytic surface and \mathfrak{g} a Lie algebra of analytic vector fields on M that are tangent to ∂M . If $X \in \mathfrak{g}$ spans a one-dimensional ideal, then:

(a) $Z(\mathfrak{g})$ meets every essential block K of zeros for X ,

(b) if M is compact and $\chi(M) \neq 0$ then $Z(\mathfrak{g}) \neq \emptyset$.

The hypotheses imply $\mathfrak{g} \subset W(X)$, because if $Y \in \mathfrak{g}$ then $[X, Y] = cX, c \in \mathbb{R}$. Therefore (a) follows from Theorem 1.9. Conclusion (b) is a consequence, because its assumptions imply the block $Z(X)$ is essential for X (Proposition 3.6). ■

Proof of Theorem 1.11

An effective analytic action α of G on M induces an isomorphism ϕ mapping the Lie algebra \mathfrak{g}_0 of G isomorphically onto a subalgebra $\mathfrak{g} \subset \mathcal{V}^\omega(M)$. Let $X^0 \in \mathfrak{g}_0$ span the Lie algebra of a one-dimensional normal subgroup of G . Then $\phi(X^0)$ spans a 1-dimensional ideal in \mathfrak{g} , hence Theorem 1.10 implies $Z(\mathfrak{g}) \neq \emptyset$. The conclusion follows because $Z(\mathfrak{g}) = \text{Fix}(\alpha(G))$. ■

References

- [1] M. Belliard, *Actions sans points fixes sur les surfaces compactes*, Math. Zeit. **225** (1997) 453–465
- [2] M. Belliard & I. Liousse, *Actions affines sur les surfaces*, Publications IRMA, Universite de Lille, **38** (1996) exposé X
- [3] C. Bonatti, *Champs de vecteurs analytiques commutants, en dimension 3 ou 4: existence de zéros communs*, Bol. Soc. Brasil. Mat. (N.S.) **22** (1992) 215–247
- [4] A. Dold, *Fixed point index and fixed point theorem for Euclidean neighborhood retracts*, Topology **4** (1965) 1–8
- [5] A. Dold, “Lectures on Algebraic Topology,” Die Grundlehren der mathematischen Wissenschaften Bd. 52, second edition. Springer, New York (1972)
- [6] D. Epstein & W. Thurston, *Transformation groups and natural bundles*, Proc. London Math. Soc. **38** (1979) 219–236
- [7] M. Freedman, *The topology of 4-dimensional manifolds*, J. Diff. Geometry **17** (1982) 357–453
- [8] S. Friedl, I. Hambleton, P. Melvin, P. Teichner, *Non-smoothable four-manifolds with infinite cyclic fundamental group*, Int. Math. Res. Not. 2007, Vol. 2007, article ID rnm031, 20 pages, doi:10.1093/imrn/rnm031
- [9] P. Hartman, “Ordinary Differential Equations,” John Wiley & Sons, New York (1964)
- [10] M. Hirsch, “Differential Topology,” Springer-Verlag, New York, 1976
- [11] M. Hirsch, manuscript (2003)
- [12] M. Hirsch, *Actions of Lie groups and Lie algebras on manifolds*, “A Celebration of the Mathematical Legacy of Raoul Bott,” 69–78. Centre de

- Recherches Mathématiques, U. de Montréal. Proceedings & Lecture Notes
Vol. 50 (P. R. Kotiuga, ed.), Amer. Math. Soc. Providence RI (2010)
- [13] M. Hirsch, *Smooth actions of Lie groups and Lie algebras on manifolds*, J. Fixed Point Th. App. **10** (2011) 219–232
 - [14] M. Hirsch & A. Weinstein, *Fixed points of analytic actions of supersoluble Lie groups on compact surfaces*, Ergod. Th. Dyn. Sys. **21** (2001) 1783–1787
 - [15] H. Hopf, *Vektorfelder in Mannifaltigkeiten*, Math. Annalen **95** (1925), 340–367
 - [16] N. Jacobson, “Lie Algebras,” John Wiley & Sons, New York (1962)
 - [17] R. Kirby, Personal communication (2012)
 - [18] E. Lima, *Common singularities of commuting vector fields on 2-manifolds*, Comment. Math. Helv. **39** (1964) 97–110
 - [19] W. Johnson, J. Lindenstrauss, G. Schechtman, *Extensions of Lipschitz maps into Banach spaces*, Israel J. Math. **54** (1986) 129–138
 - [20] S. Łojasiewicz, *Triangulation of semi-analytic sets*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964) 449–474
 - [21] P. Molino & F.-J. Turiel, *Une observation sur les actions de \mathbf{R}^p sur les variétés compactes de caractéristique non nulle*, Comment. Math. Helv. **61** (1986), 370–375
 - [22] P. Molino & F.-J. Turiel, *Dimension des orbites d'une action de \mathbf{R}^p sur une variété compacte*, Comment. Math. Helv. **63** (1988) 253–258
 - [23] G. Mostow, *The extensibility of local Lie groups of transformations and groups on surfaces*, Ann. of Math. **52** (1950) 606–636.
 - [24] E. Nelson, “Topics in Dynamics I: Flows,” Princeton University Press (1970)
 - [25] J. Plante, *Fixed points of Lie group actions on surfaces*, Erg. Th. Dyn. Sys. **6** (1986) 149–161
 - [26] J. Plante, *Lie algebras of vector fields which vanish at a point*, J. London Math. Soc (2) **38** (1988) 379–384
 - [27] J. Plante, *Elementary zeros of Lie algebras of vector fields*, Topology **30** (1991) 215–222
 - [28] H. Poincaré, *Sur les courbes définies par une équation différentielle*, J. Math. Pures Appl. **1** (1885) 167–244

- [29] A. J. Schwartz, *Generalization of a Poincaré-Bendixson theorem to closed two dimensional manifolds*, Amer. J. Math. **85** (1963) 453-458
- [30] R. Sacksteder, *Foliations and pseudogroups*, Amer. J. Math. **87** (1965) 79–102
- [31] W. Thurston, *A generalization of the Reeb stability theorem*, Topology **13** (1974) 347-352
- [32] F.-J. Turiel, *An elementary proof of a Lima's theorem for surfaces*, Publ. Mat. **3** (1989) 555–557
- [33] F.-J. Turiel, *Analytic actions on compact surfaces and fixed points*, Manuscripta Mathematica **110** (2003) 195–201
- [34] F.-J. Turiel, personal communication (2006)
- [35] H. Whitney, *Regular families of curves*, Ann. Math. **34** (1933) 244–270